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Technical Report No. 399

Conditions for Optimum Digital Communication with  
Application to Delta Modulation

By Terrence Fine

March 5, 1963

Contract Nonr-1866(16)/NR-372-012

Division of Engineering and Applied Physics

Cruft Laboratory, Harvard University

Cambridge, Massachusetts

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### ERRATA

Page   Line

- 12      11      Eq. 3 should read Eq. 4 .
- 15      10      ... unknown, but affirmative answer is expected.  
should read ... unknown.
- 30      9      For  $R_n(j, i_p)$  read  $R_n(j)$  .
- 32      last      The words ... reconstruction of should be followed by:  
 $I(t_n)$  by  $R(t_n)$  .
- 35      last      For  $S_2 = (-4, -2; 1)$  read  $S_2 = (-5, -2, 1)$  .
- 36      6      For Ave. of Both =  $(-1.25)$  read Ave. of Both =  $(-1.75)$  .
- 36      12      For ... from  $(-4)$  to  $(-2.5)$  read ... from  $(-5)$  to  $(-1)$  .
- 36      last      For  $(-2)$  read  $(-2.5)$  .
- 38      18      Note: Equation 7 is only an approximation obtained by  
assuming an error free estimation of level sets. Further  
work currently in progress considers level set errors in  
addition to the others, and it indicates that the inverse  
dependence upon  $J$  given by Eq. 9 is optimistic. The error  
probably decreases more slowly with increasing sample size.

38      last      For  $\sum_{i,k}^J E(I_n)_i$  read  $\sum_{i,k}^J E((I_n)_i) \dots$  .

May 3, 1963

Office of Naval Research

Contract Nonr-1866(16)

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Technical Report

on

CONDITIONS FOR OPTIMUM DIGITAL COMMUNICATION  
WITH APPLICATION TO DELTA MODULATION

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## ABSTRACT

↓  
A procedure for optimisation of a class of binary communication systems is presented and applied. The message set or transmitter input is taken to be a real-valued sample sequence from a stochastic process with discrete parameter. The transmitter may be any time-varying, nonlinear operator with domain of the real valued input and range to the binary numbers. The transmission medium or noisy channel linking the transmitter and receiver is to be characterized by the conditional probabilities of all possible received binary sequences given any transmitted sequence. The receiver may be any real-valued, time-varying, nonlinear operator on the received binary sequences. ~~The requirements of the message destination are incorporated in a system~~ error criterion, which is taken as the statistical average of a loss function,  $\phi$ , of the difference between transmitter input and receiver output; the function  $\phi$  is always taken as almost everywhere differentiable; and additional assumptions such as strictly increasing for increasing magnitude of its argument and bounded away from minus infinity, are imposed when they lead to interesting results. This restricted class of functions  $\phi$  will be denoted by  $\Phi$ .

Cont'd  
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The optimisation procedure we develop is much like one proposed by S. P. Lloyd of Bell Telephone Laboratories for use in noiseless optimum quantisation schemes and can be so applied. Results, consisting of necessary conditions that the transmitter and receiver or quantiser and interpolator must satisfy, can be obtained for many members of the family of loss functions,  $\Phi$ , referred to above. In particular, the choice of a quadratic loss,  $\phi(x) = x^2$ , is explored in some detail.

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The optimisation conditions obtained are discussed and the relationship between an optimum communication system and a delta modulation system indicated. *It is shown* ~~We are able to show~~ that, for the quadratic loss function and any noisy channel, a delta modulation system is an allowable representation of the optimum binary system. ~~Furthermore, if we restrict ourselves to a noiseless channel,~~ we can prove that the above conclusion also obtains for the class  $\Phi$  of loss functions,  $\phi$ , satisfying

$$\phi(z_1) > \phi(z_2) > -\infty \text{ if and only if } |z_1| > |z_2|$$

Explicit design examples are then provided in addition to a discussion of the possibilities of automatic design via application of digital computers.

~~We conclude by discussing~~ preliminary results on an extended digital communication system model <sup>are discussed</sup> and ~~indicating~~ areas for further research, <sup>are</sup> indicated.

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**CONDITIONS FOR OPTIMUM DIGITAL COMMUNICATION  
WITH APPLICATION TO DELTA MODULATION**

by

**Terrence Fine**

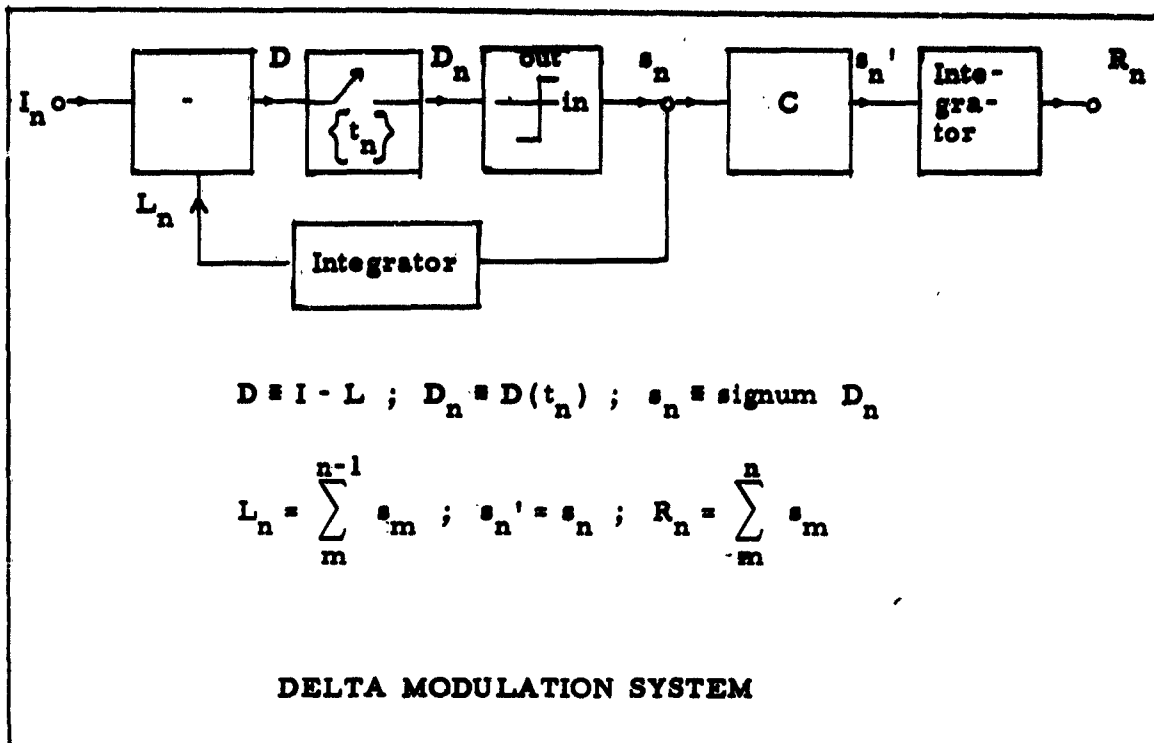
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**I. INTRODUCTION AND BACKGROUND**

This report attempts to present a method of digital system optimization by development and application. The optimum communication system, which we shall synthesize by the determination of necessary conditions for design, will be seen to possess, under certain restrictions, a representation as a delta modulation system. Before proceeding with a more precise approach to this problem, let us establish a motivational background by considering delta modulation.

The delta modulation system as proposed in 1946 is sketched in the block diagram below. The transmitter possessed a feedback loop which integrated the binary transmitter output and compared the result with the current input. According as the input,  $I_n$ , at the input time,  $t_n$ , was greater or less than the feedback value,  $L_n$ , the transmitted output,  $s_n$ , was in one or the other of its two states. For convenience, the two transmitter output states will be taken as plus or minus unity, respectively.





The channel, which was taken to be noiseless, provided the input to a receiver which then integrated this input to generate its output. Thus, the receiver output was similar to that of the transmitter feedback loop, and the transmitter output served to correct the receiver output insofar as it exceeded or fell below the transmitter input. A common supplement to the receiver indicated above was to follow its output with a band-pass filter of bandwidth equal to that of the input message set.

The system described above, taken together with the one modification of replacing the single integrator by a double integrator to achieve a better tracking response, represents the design state of the art until recently.

Optimization adjustments with respect to intelligibility or noise power criteria were made for the most part by experiment. The existing analysis was either very qualitative in nature or attempted to determine properties of the error spectrum for the single or double integrating modulator. Mention of some of the representative work in this domain will be found in References 2 through 6.

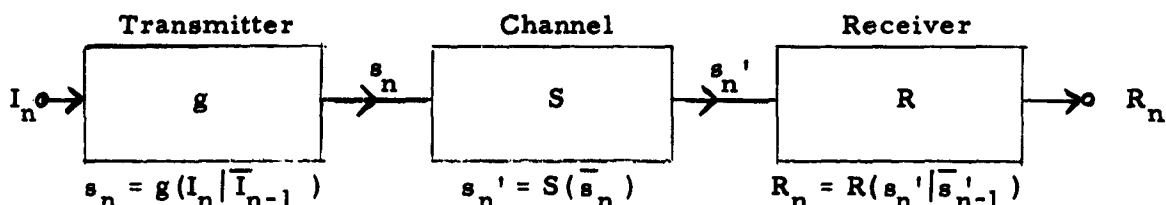
Experimental results that were available indicated that the delta modulator for speech inputs was inferior to pulse code modulation at moderate levels of fidelity and above. This led us to consider the possibility of generalizing the current system and optimizing the more general system assumed in order to make it superior to pulse code modulation. The variations that were attempted were to remove the constraints that the receiver must be like the feedback loop and that both of these devices need be integrators or even linear operators. We permitted the receiver and feedback loop to contain general nonlinear operators, including integrators as special cases. Then we successfully attempted to derive the relations governing the optimum selection of these functionals with respect to some class of criteria.

In this report we should like to move the optimization problem one step further back by removing the restriction that the transmitter be representable by a feedback configuration. We should then like to determine the conditions under which such a configuration will be optimum and what its properties must be, if it is optimum. We shall also take the communications channel to be noisy and later specialize to the noiseless case. The results we derive will be in the form of necessary conditions that the components of the optimum communication system must satisfy, and the requirements for sufficiency will be only implicitly stated.

It should be mentioned that our results were partially anticipated by S. P. Lloyd [1] and J. Max [7] in connection with the quantization problem. The approach we employ was motivated and obtained independently from that of Lloyd, but was first stated by him in a similar form. We have learned from Lloyd's unpublished exposition.

## II. SYSTEM MODEL

A precise description of the binary communications system we wish to optimize and discuss is summarized in the diagram below, and explicated in this section. Possible extensions of this model will be considered in Section IX.



With reference to the figure given above, we take for the input message or signal set (represented by  $I$ ) a stochastic process with a discrete time parameter,  $t_n$ , stating the time of occurrence of a real valued input random variable,  $I_n$ . The multivariate probability distribution functions that describe this stochastic process are assumed known.

The transmitter operates on a sequence of message inputs,  $\bar{I}_n$ , to produce a binary output sequence  $\bar{s}_m$ . A superscript bar denotes a sequence extending into the past with most recent member indicated by the subscript. For example:

$$\bar{I}_n = \{ I_n, I_{n-1}, \dots, I_{n-i+1} \} \quad (1)$$

where the positive integer  $i$  may be arbitrarily large.

The requirement that a communication system be binary implies that the transmitter output sequence,  $\bar{s}_m$ , be composed only of binary valued elements, which for definiteness we take to be plus or minus unity. It is

important to note that the output binary sequence is of the same length (same subscript) as the input sequence; without loss of generality we may neglect consideration of the physical delay in the transmitter operation.

We represent the transmitter by some function  $g$  with domain equal to the range of the input  $I_n$  and range consisting of only plus or minus one, or  $s_n$ . The function  $g$  will depend upon the prior input sequence,  $\bar{I}_{n-1}$ , for its explicit form, this being an alternative statement of its functional dependence upon the entire past of the input.

$$s_n = g(I_n | \bar{I}_{n-1}) \quad (2)$$

The point to be emphasized is that upon the receipt of the input  $I_n$  a binary output  $s_n$  is generated in a manner which depends upon the past history of the input process.

In an actual system, the output of the transmitter described above would serve as the input to a pulse modulator whose output would in turn be fed into a communications channel. At the receiver a distorted version of the modulator output, produced by random and deterministic characteristics of the transmission medium, is transformed by the demodulator into another binary sequence,  $\bar{s}_n'$ . The effects of the deterministic channel distortions may, in principle, be completely eliminated, leaving only the random distortions of the so-called noisy channel. In general, the terminal characteristics of such a noisy channel can be described by the conditional probability function for an output sequence,  $\bar{s}_n'$ , given an input sequence,  $\bar{s}_n$ , and we denote it by  $P(\bar{s}_n' | \bar{s}_n)$ .

The receiver operates on the received sequence,  $\bar{s}_n'$ , to produce a real valued output sequence  $\bar{R}_n$ . We assume that these sequences are of equal length or have the same subscript. As before, there is no loss of generality attendant upon the neglect of physical time delays in the actual system.

### III. STUDY OBJECTIVE AND FORMULATION

To design the very general binary communication system just described so as to optimize its performance in some manner, requires a stated performance or system error criterion. Such a criterion maps any transmitter and receiver functional pair into a real number, and the convention is adopted that smaller numbers represent preferable systems. An appropriate and generally useful class of criteria employs the difference between system input and output as the error variable and should be used to judge the overall system, rather than its individual components. The error variable,  $I_n - R_n$ , is then weighted by means of a loss function  $\phi(I_n - R_n)$  chosen in accordance with some preconceived notions as to what constitutes a significant characteristic of behavior (e. g., we may lightly weight small errors and heavily penalize large ones).

By far the most common selection for  $\phi(x)$  is the quadratic function  $x^2$  and a distant second might be  $|x|$ . However, for the moment we take  $\phi$  to be almost everywhere differentiable and bounded away from minus infinity. Finally, we observe that, by the hypothesis of a random process input and a noisy channel, the input and output and, therefore, their difference, is a random variable. This requires us to average  $\phi(I_n - R_n)$  over all inputs and noisy channel outputs to obtain an overall error measure or a mapping of transmitter and receiver independent of specific inputs and channel noise.

System error =  $E(\phi(I_n - R_n))$  where  $E(\cdot)$  is the statistical expectation;  $\phi$  must be chosen so that this quantity exists finitely.

For any particular error criterion in the above class we wish to determine the optimum transmitter and receiver, given the statistical description of the message process and the channel. In order to accomplish this objective, we will first represent the transmitter and receiver more explicitly.

In particular, we are interested in the inverse image of a plus one transmission ( $s_n = 1$ ) with respect to the mapping induced by the transmitter function,  $g$ . From Eq. 1 of the previous section the inverse image is a set of input values,  $\ell_n^+$ , given by

$$\ell_n^+(\bar{I}_{n-1}) = \left\{ I_n : g(I_n | \bar{I}_{n-1}) = +1 \right\}.$$

The design of a transmitter corresponds to a selection for the point set  $\ell_n^+$  with the convention that we transmit a plus one, if  $I_n$  falls in  $\ell_n^+$ , and a minus one, if  $I_n$  lies in  $\ell_n^-$ , the complement of  $\ell_n^+$  with respect to the real line.

The receiver design consists of the specification of a real number for each possible received sequence,  $\bar{s}_n'$ , although only the most recent members of the received sequence may actually affect the choice of a receiver output; we then say the receiver has a finite memory. If we assume that the receiver possesses a finite memory of length  $N$ , then we need only specify  $2^N$  receiver outputs,  $R_n(\bar{s}_n')$ . It may be noted that using a finite receiver memory requires the specification of initial conditions to make the problem physically definite, but we need not explicitly evoke such a set of conditions in our present work. Finally, we take the memories of the transmitter and receiver to be of equal length ( $i$  equals  $N$  in Eq. 1 of the second section).



## IV. OPTIMIZATION PROCEDURE

In this section we shall determine necessary conditions that  $l_n^+$  and the set  $\{R_n\}$  must satisfy, if they are to yield a minimum system error. Lack of sufficiency will arise, not from the non-existence of a minimum for the system error, but from the possibility of several solutions to the necessary conditions. Sufficiency is then obtained by checking all of the solutions to the necessary conditions by means of the value of  $E(\phi(I_n - R_n))$ , or the system error, that they give rise to, and then choosing that solution yielding the minimum overall error.

The development of the necessary conditions for an optimum design proceeds in three stages: (1) Assume a set of receiver outputs  $R_n(\bar{s}_n')$  as being given and find the optimum point set  $l_n^+$  for this choice. (2) Assume the point set  $l_n^+$  as known (and, therefore,  $l_n^-$ , also) and find the optimum receiver outputs corresponding to this transmitter design. (3) Using the interrelated system design equations of (1) and (2), solve them simultaneously to find the jointly optimum system design. This general procedure is quite analogous to the minimization of a function of several variables in the calculus; the optimum value of each variable assuming all the others constant is found, and then all of the conditions are solved simultaneously for the actual minimum or stationary point.

The point set  $l_n^+$  is determined by the requirement that the error resulting from transmission of a plus one should be less than or equal to that resulting from transmission of a minus one whenever  $I_n$  falls in the set  $l_n^+$ . The transmitter is aware of the actual value of the input, but possesses only imperfect knowledge as to the effect of a transmission at the receiver; the

channel noise acts to render the receiver output into a random variable with a distribution dependent upon the transmitted sequence. Thus, the error we wish to minimize is an error averaged over all of the possible, initially given, receiver outputs. Our discussion is then summarized by

$$I_n \varepsilon l_n^+ \iff E[\phi(I_n - R_n(\bar{s}_n')) | \bar{s}_n^+] \leq E[\phi(I_n - R_n(\bar{s}_n')) | \bar{s}_n^-] \quad (1)$$

where

$$\bar{s}_n^+ = \left\{ s_n = 1, \bar{s}_{n-1} \right\}, \quad \bar{s}_n^- = \left\{ s_n = -1, \bar{s}_{n-1} \right\}.$$

We digress briefly to introduce a representation for the set  $l_n^+$  as a union or sum (as the components are disjoint) of closed intervals, with closure being guaranteed by the presence of the equality sign in the above inequality.

$$l_n^+ = \bigcup_i [L_{oi}, L_{vi}] \text{ with } L_{oi} \leq L_{vi} \quad (2)$$

We should like to find a necessary condition that the end points of the intervals in the above union must satisfy. Observe that  $l_n^-$ , being the complement of  $l_n^+$ , consists of the open intervals adjacent to the closed intervals of  $l_n^+$ . Thus, for inputs slightly below the left end point of any interval in  $l_n^+$  or slightly above the right end point of any interval in  $l_n^+$  we have that a minus one transmission is optimum or that the direction of the inequality in Eq. 1 is reversed. The assumed continuity of  $\phi$  and the reversal of the inequality as the input is varied about the end points assure us that the end points of the intervals in either  $l_n^+$  or  $l_n^-$  (and, excepting  $\pm \infty$ , they are the same) must be those real input values for which equality obtains in Eq. 1! More concisely, we have

$$E[\phi(L - R_n(\bar{s}_n')) | \bar{s}_n^+] = E[\phi(L - R_n(\bar{s}_n')) | \bar{s}_n^-] \text{ with } L = L_{oi} \text{ or } L_{vi} \quad (3)$$

Not all of the real roots of the above equation necessarily correspond to the end points of an  $I_n$  interval, for only those points of equality that mark a transition in the direction of the inequality 1 are of significance; this is, then, only a necessary condition for the optimum transmitter given the receiver. If both sides of the inequality 1 evaluated at the point  $\underline{L}$ , which is a solution of Eq. 3, do not have the same slope, or derivative with respect to  $I_n$  at the point  $\underline{L}$ , then the direction of the inequality 1 undergoes a reversal at the point where  $I_n$  equals  $\underline{L}$ . Thus, a condition that guarantees a transition in the inequality for a root  $\underline{L}$  of Eq. 3 is

$$E(\phi'(L - R_n(\bar{s}_n')) | \bar{s}_n^+) \neq E(\phi'(L - R_n(\bar{s}_n')) | \bar{s}_n^-). \quad (4)$$

We should mention that the new condition provided by Eq. 3 is sufficient to guarantee a change in the direction of the inequality 1, but not necessary; equality may obtain and  $\underline{L}$  still be a point of transition, because it is also a point of inflection. We shall discuss the properties of these results more fully in the next section.

If we now fix the transmitter design and turn to the receiver, we note that the receiver knowing the input statistics, the transmitter design, the channel noise statistics, and the received sequence can determine the distribution function of the input conditional upon the received binary sequence. The receiver must then process this distribution function to yield that number which results in a minimum system error. That is, we must:

$$\underset{R_n}{\text{minimize}} \quad E(\phi(I_n - R_n) | \bar{s}_n') \quad (5)$$

The conditional expectation has an integral representation employing the conditional distribution function for the input given the received sequence.

A necessary condition that the number  $R_n$  must satisfy to minimize the integral can be obtained from the calculus and is:

$$\frac{\partial}{\partial R_n} \int_{-\infty}^{\infty} \phi(x - R_n) dP[x > I_n | \bar{s}_n'] = 0 \quad (6)$$

Interchanging orders of differentiation and integration, we have:

$$\int_{-\infty}^{\infty} \phi'(x - R_n) dP[x > I_n | \bar{s}_n'] = 0 \quad (7)$$

The above equation rewritten more compactly yields the necessary condition that determines the optimum receiver for a fixed transmitter.

$$E[\phi'(I_n - R_n) | \bar{s}_n'] = 0 \quad (8)$$

The final step in an actual optimization problem would be to search through the solutions of both of the derived necessary conditions Eqs. 3 and 8 to find that pair resulting in the absolute minimum system error. It may be possible to solve this problem in a closed analytic fashion, but this is generally the exception. More commonly, we would be driven to some iterative procedure.

In the next two sections we shall indicate some conditions under which there is only one real  $L$  solution to Eq. 3;  $L_n^+$  is then a single semi-infinite interval. In such a practically important case, the iterative procedure might proceed by first assuming a trial value of  $L$  and using it in Eq. 8 to determine the various receiver outputs. With the calculated receiver outputs we can turn to Eq. 3 and determine a check value of  $L$ . If the check value is lower than the trial value of  $L$ , then the next trial value should be decreased from its original magnitude and vice versa. This particular iterative procedure is only meant as an illustration and is only known to be applicable in the examples to be discussed shortly where  $L$  is unique.

## V. IMPLICATIONS OF THE DERIVED NECESSARY CONDITIONS

Equation 3 for the transmitter design depends upon the actual input process only insofar as  $\bar{s}_{n-1}$  is concerned and in no other way. We have the direct conclusion that the end points of the intervals making up  $\bar{I}_n$  depend only upon  $\bar{s}_{n-1}$ ; therefore,  $I_n$  is a functional only of  $\bar{s}_{n-1}$ , and, finally,  $g$  is also only dependent upon the past of the transmitted sequence. This argument produces the interesting conclusion that the optimum transmitter does not utilize the fully available past of the input,  $\bar{I}_{n-1}$ , but only the transmitter output derived from that input sequence,  $\bar{s}_{n-1}$ !

If for the loss function,  $\phi$ , we use the very common quadratic function, we can rewrite Eq. 3 by utilizing the observation that in Eq. 3  $L$  is just a real number and can commute with the conditional expectation operator. This leads to

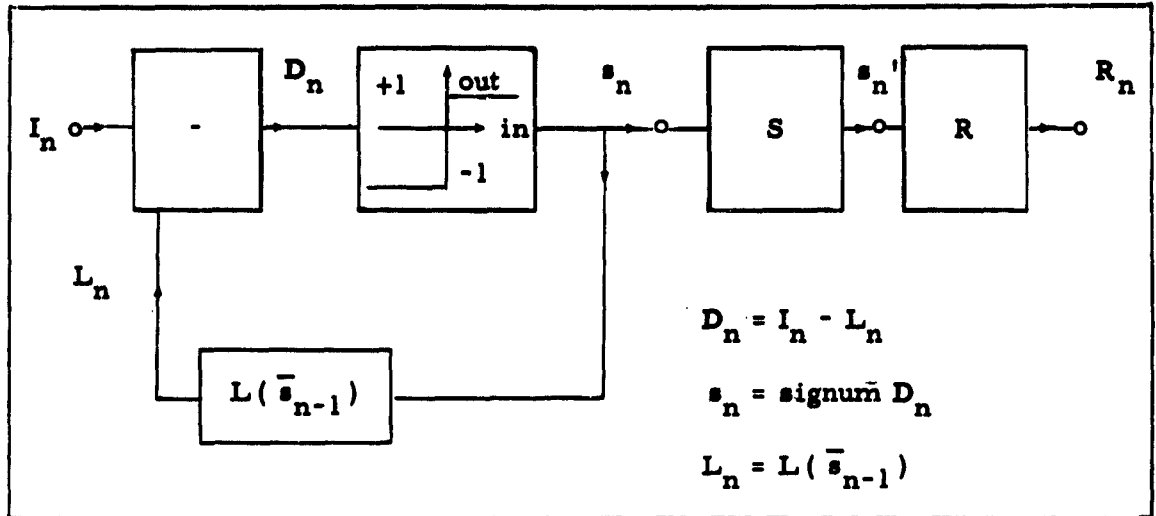
$$L^2 - 2LE[R_n(\bar{s}_n') | \bar{s}_n^+] + E[R_n^2 | \bar{s}_n^+] = L^2 - 2LE[R_n(\bar{s}_n') | \bar{s}_n^-] + E[R_n^2 | \bar{s}_n^-] \quad (1)$$

Eliminating the  $L^2$  terms and solving the resulting linear equation leads to the following explicit expression for  $L$ :

$$L(\bar{s}_{n-1}) = \frac{1}{2} \cdot \frac{E[R_n^2 | \bar{s}_n^+] - E[R_n^2 | \bar{s}_n^-]}{E[R_n | \bar{s}_n^+] - E[R_n | \bar{s}_n^-]} \quad (2)$$

The significance of Eq. 2 is that for any assumed receiver outputs there is one and only one solution to Eq. 3;  $I_n$  is just a semi-infinite interval.

The above conclusions for the quadratic loss function lead us to the realization that an alternative block diagram for our binary communications system can be given utilizing a feedback loop.



If  $I_n$  falls in the semi-infinite interval with lower end point of  $L_n$ , then  $D_n$  is positive,  $s_n$  is plus one, and we are classifying the inputs by means of the proper form of point set,  $\ell_n^+$ . The feedback loop provides precisely the  $\bar{s}_{n-1}$  needed to generate correctly the lower end point of  $\ell_n^+$ .

This feedback device is the generalization of the delta modulation system investigated by others, and our present work shows that the delta modulation configuration leads to a quadratically optimal, binary communication system in the presence of an arbitrarily noisy channel! Whether or not this result (of a unique solution to Eq. 3 of the previous section) holds for a variety of loss functions is at present unknown, but an affirmative answer is expected.

The necessary condition for an optimal receiver given by Eq. 8 of the previous section is a reasonable one. The information at the receiver concerning transmitter input and channel noise statistics as well as the transmitter design and the received binary sequence enable it to determine the conditional

probability of the input given all of this information. The receiver then selects the "best" single estimate of the input from this distribution and the given loss function. If the loss function is quadratic, one can readily show that the receiver operates by taking the conditional expectation of the input, given the received sequence as its output; the familiar mean as the solution to the minimum variance problem. In a similar vein, if the loss function is taken as the absolute value of the error variable, then we find that the optimum receiver has as its output the median of the conditional distribution of the system input given the received binary sequence. This result is also well known.

## VI. NOISELESS CHANNEL

An important specialization of our results is to the case of the noiseless channel or the channel in which the received binary sequence is identical with the transmitted sequence,  $\bar{s}_n$ . The transmitter design relation can now be appreciably simplified by the observation that the receiver output given the transmitted sequence is a degenerate random variable or constant, and, therefore, the expectation is superfluous. Equation 3 of the optimization section reduces to:

$$\phi(L - R_n(\bar{s}_n^+)) = \phi(L - R_n(\bar{s}_n^-)) \quad (1)$$

If we generalize from the quadratic and absolute value loss functions to a class of  $\phi$ ,  $\Phi$ , which satisfies

$$\phi \text{ a.e. differentiable; } |z_1| > |z_2| \iff \phi(z_1) > \phi(z_2) > -\infty \quad (2)$$

then the unique solution of Eq. 1 becomes:

$$L = \frac{1}{2}R_n(\bar{s}_n^+) + \frac{1}{2}R_n(\bar{s}_n^-) = L(\bar{s}_{n-1}) \quad (3)$$

Equation 3 can be verified by use of Eq. 2 and substitution of Eq. 3 in Eq. 1; uniqueness follows from the strict inequality in Eq. 2.

The essential point is that, for the noiseless channel and an important class of loss functions, classification of the input, at a particular time, is only based upon whether it is larger or smaller than a particular number,  $L$ . This is the same conclusion that we reached for the quadratic criterion and the noisy channel, and, therefore, the representation of the optimum system as a delta modulation system is again possible.

The specialization of the receiver relation is trivial and is simply:

$$E(\phi'(I_n - R_n) | \bar{s}_n) = 0 \quad (4)$$



## VII. CALCULATED DESIGN EXAMPLES (NOISELESS)

Equations 3 and 4 of Section VI provide us with conditions that must be satisfied by the optimum transmitter and receiver, respectively. These equations are not difficult to implement, if we are provided with an analytically tractable conditional distribution or density function for the input process. Unfortunately, it is far from common that the multivariable distribution functions for an input stochastic process will be given in terms of finite combinations of elementary functions, and this makes calculations with such functions laborious. Two exceptions to this dictum will be examined below, and they represent opposite ends of the range for dependence between members of a sequence. The loss function  $\phi(x)$  will be taken to be quadratic or  $x^2$ .

### Independence

We first assume that the input stochastic process consists of a sequence of independent, identically distributed random variables with common density function  $p(I)$ . Then from Eq. 4 and the quadratic loss we have, after differentiation and separation of terms

$$R_n(s_n = 1) = E(I_n | s_n = 1) ; \quad R_n(s_n = -1) = E(I_n | s_n = -1) . \quad (1)$$

The receiver output should be the conditional mean or the mean of the input distribution as known at the receiver. The mutual independence of members of the input sequence and the given density function allow us to rewrite Eq. 1 by employing the integral representation for the expectation

$$R_n(s_n = 1) = \frac{\int_{L_n}^{\infty} xp(x) dx}{\int_{L_n}^{\infty} p(x) dx} ; \quad R_n(s_n = -1) = \frac{\int_{-\infty}^{L_n} xp(x) dx}{\int_{-\infty}^{L_n} p(x) dx} \quad (2)$$

$L_n$  , is the as yet unknown position of the transmitter feedback loop output, and it can be found from Eq. 2 above and Eq. 3 of Section VI as follows:

$$2L_n = \frac{\int_{L_n}^{\infty} xp(x) dx}{\int_{L_n}^{\infty} p(x) dx} + \frac{\int_{-\infty}^{L_n} xp(x) dx}{\int_{-\infty}^{L_n} p(x) dx} \quad (3)$$

If we have a specific  $p(I)$  in mind, we may proceed to solve the transcendental equation given by Eq. 3 for  $L_n$  and then use  $L_n$  in Eq. 2 to determine the two receiver outputs. For example, if  $p(I)$  is the exponential distribution for positive  $I$  , the integration can be carried out, and Eq. 3 yields:

$$\exp(-L_n) = 1 - \frac{L_n}{2} \quad \text{or} \quad L_n \doteq 1.6 .$$

Having found the unique setting of the feedback loop, we may return to Eq. 2 and calculate the two receiver outputs

$$R_n(s_n = 1) = 2.6 ; \quad R_n(s_n = -1) = .6 .$$

Further calculations are straightforward and we summarize their results:

$$\text{Prob}(s_n = 1) = .2 ; \text{var } I_n = 1 ; E((I_n - R_n)^2) = .36 .$$

This completes the design of the optimum system for such an input process, and describes its performance.

### Complete Dependence

The case of complete dependence in chains of length  $M$  and independence between successive chains can be treated fairly directly, because of the relative simplicity of the resulting distribution functions. We assume that  $I_1$  through  $I_M$  are equal to some random variable  $I$  with density function  $p(I)$ . Then the transmitter and receiver design for the first member of the sequence,  $I_1$ , are precisely as given in Eqs. 2 and 3 above. As an illustration of how the problem proceeds, we give the design equations for the system concerning  $I_2$  and assuming that  $s_1 = 1$ . Differences occur in the interval of integration and in the denominator, due to the change in the conditional density function

$$p(I_2 | s_1 = 1) = \begin{cases} \frac{p(I_2)}{\int_{L_1}^{\infty} p(I) dI} & \text{if } I_2 \geq L_1 \\ 0 & \text{if } I_2 < L_1 \end{cases} \quad (4)$$

The design equations then become

$$R_2(s_2 = 1, s_1 = 1) = \frac{\int_{L_2}^{\infty} xp(x) dx}{\int_{L_2}^{\infty} p(x) dx} ; R_2(s_2 = -1, s_1 = 1) = \frac{\int_{L_1}^{L_2} xp(x) dx}{\int_{L_1}^{L_2} p(x) dx} ;$$

(5)

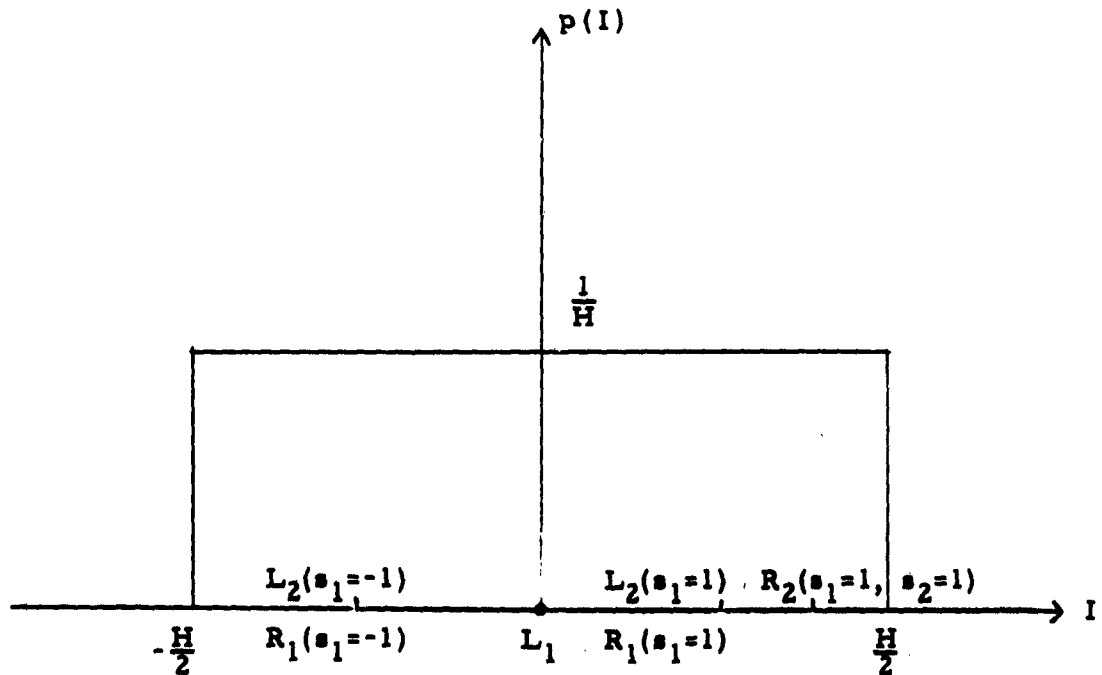
$$2L_2(s_1 = 1) = \frac{\int_{L_2}^{\infty} xp(x) dx}{\int_{L_2}^{\infty} p(x) dx} + \frac{\int_{L_1}^{L_2} xp(x) dx}{\int_{L_1}^{L_2} p(x) dx} .$$

The process just indicated is then continued until the  $M^{\text{th}}$  step, after which it recycles.

It is possible to apply Eq. 5 to the exponential distribution discussed in the case of independence. If one does so, one straightforwardly finds after integration and the solution of a simple transcendental equation:  $L_2(s_1 = 1) \doteq 3.20$ ;  $R_2(s_1 = s_2 = 1) = 4.20$  ;  $R_2(s_1 = -s_2 = 1) = 2.20$  ;  $\text{Prob}(s_2 = 1 | s_1 = 1) = .203$ ;  $\text{var } I_2 = 1$  ;  $E((I_2 - R_2)^2 | s_1 = 1) = .32$  . We notice that the second step provides little improvement in performance over the first step, under the hypothesis that  $s_1 = 1$  . The continuation of this example to investigate the other alternatives is simple, but involves repetitive elementary calculations.

As another illustration we might consider that the density function,  $p(I)$  , was uniform over some range  $H$  . We would then find that  $L_1$  occurred at the midpoint of the range, and the two receiver outputs,  $R_1(s_1 = 1)$  and  $R_1(s_1 = -1)$  , occurred at the midpoints of the right and left halves or at

the  $3/4$  and  $1/4$  points of the full range, respectively. The transmitter continues operation by successively halving the remaining range into which  $\underline{I}$  is known to have fallen, and the receiver output is taken at the midpoint of the reduced range. Finally, the  $M$ th digit asserts that  $\underline{I}$  is in some interval of length  $H/2^M$  and that the receiver output is at the midpoint of this interval. The system performance may then be readily calculated at each step, and we summarize the results that obtain just after the  $j$ th binary digit has been received:  $\text{var } I_j = H^2/12$ ;  $E((I_j - R_j)^2 | \bar{s}_j) = H^2 / 12 \cdot 2^{2j}$ ;  $\text{Prob}(s_{j+1} | \bar{s}_j) = 1/2$ . These results for  $j$  equal to unity also provide the solution to the case of independence between inputs when the input distribution is uniform.



The case of dependence that we have discussed above corresponds to a process of  $M$ -bit pulse code modulation (PCM) . The input value  $\underline{I}$  is represented by a binary sequence  $\underline{M}$  in length, from which its magnitude can be reconstructed. This particular quantization procedure is sometimes referred to as programmed PCM .

## VIII. COMPLEMENTS FOR DELTA MODULATION

The results described in the previous sections have been examined in greater detail than hitherto presented, particularly with regard to the properties of a delta modulation representation and a noiseless channel. In this section we should like to discuss briefly two results that follow from our analysis, as well as to indicate our application of a digital computer to obtain additional conclusions.

We notice from Eq. 3 of Section VI that the optimum transmitter feedback loop is linearly related to the chosen receiver. If for various reasons, including engineering practicality, we wish to select a linear operator on the received binary sequence, or, alternatively stated, a linear time-varying system, for the receiver, then we can show that the optimum transmitter feedback loop should be chosen to be the same operator restricted to the past of the received sequence. To prove this statement, take the following representation for the receiver:

$$R_n(\bar{s}_n) = \sum_m^n a_{nm} s_m \quad (1)$$

Now the expression for the optimum transmitter feedback loop operator,  $L_n(\bar{s}_{n-1})$ , is given by:

$$2L_n(\bar{s}_{n-1}) = R_n(s_n = 1, \bar{s}_{n-1}) + R_n(s_n = -1, \bar{s}_{n-1}) \quad (2)$$

Substitution of Eq. 1 into Eq. 2 yields the optimum solution for  $L_n$ .

$$L_n(\bar{s}_{n-1}) = \sum_m^{n-1} a_{nm} s_m \quad (3)$$

Equation 3 proves our conjecture concerning the relation of the optimum transmitter to the receiver constrained to be a linear operator.

Equation 3 also provides a justification, in part, for the assumption made by the early designers of delta modulation systems that the feedback loop element should be the same as the receiver. The subsequent refinement by the addition of a band-pass filter at the output of the receiver was motivated by an attempt to interpolate between samples and does not appear in our discrete time analysis, which is concerned only with the samples themselves.

A further result is an approximation to the transmitter design equation based upon the hypothesis that at a given time a positive transmission is as likely as a negative one. This assumption implies that we may rewrite Eq. 2 above as follows:

$$L_n(\bar{s}_{n-1}) = E(R_n(\bar{s}_n) | \bar{s}_{n-1}) \quad (4)$$

If, in addition, we assume a quadratic loss function so that the optimum receiver is a conditional expectation itself, then Eq. 4 becomes

$$L_n(\bar{s}_{n-1}) = E(I_n | \bar{s}_{n-1}) \quad (5)$$

The significance of Eq. 5 is that the transmitter design equation no longer depends upon the receiver design, and we may solve directly rather than iteratively. Thus, by assuming a property of the resulting optimum system, which is known not to be universally valid, we have simplified the analysis leading to the optimum design. This approximation is essentially that we will transmit at the maximum information rate with an optimum binary communication system as we have defined it; this statement, however, is only occasionally correct, and its truth depends upon the nature of the input process.



The maximum information rate approximation referred to above implies a very special structure for the input stochastic process, if it is to hold exactly. In detail it requires that at the time in question the conditional mean of the input distribution be equal to the conditional median of the input distribution. If the probability distribution function for the input at a given time conditional upon the past of the transmitted binary sequence is symmetric, then the conditions are met and the approximation is exact. An example of such a process would be the uniformly distributed dependent case of Section VII or any symmetrically distributed input process of independent random variables. The multivariate normal process does not satisfy the conditions we have suggested as establishing the exact validity of our suggested approximation. However, computer studies of selected exponentially correlated and band limited normal processes indicated that the error made in applying our approximation to the determination of  $L_n$  was less than 15%.

Computer studies based upon the results for a noiseless channel and quadratic loss function have been carried out with the following objectives: (1) Determine the feasibility of the iteration scheme outlined earlier. (2) Judge the accuracy of the maximum information rate approximation discussed above. (3) Provide an automatic design procedure for optimum binary communication systems given only input data and not input statistics. (4) Effect a simulation procedure for use in investigating system performance when the input statistics preclude reasonable analytic solution. Valuable conclusions were obtained in all of the above-mentioned areas but the last. A procedure is available, but the results of its application are inconclusive.

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The iteration scheme proved practicable on an IBM 7090 when used in conjunction with appropriate statistical estimators. Input information was provided in the form of independent sequences of fixed length and arithmetic average estimators were used to provide the needed conditional expectations. The iteration process will always converge in a finite number of steps, the number of steps being bounded by the total number of independent sequences providing the input information. This overall procedure yields a valuable means of automatically designing an optimum binary communication system directly from measurements on the message or input process. The appendix contains an illustrative example of the calculational process alluded to above.

It was also possible to estimate the optimum system for an input process with known statistics, such as multivariate normal, by first simulating the input process and then proceeding as above. The optimum system could then be tested and its performance evaluated by operating the simulated system on additional members of the message set and comparing the system output with the known input. However, the conclusions so reached have not been particularly fruitful. It is intended to pursue this work somewhat further by processing recorded quantized speech provided by the Bell Telephone Laboratories in order to determine the merits of the system we have devised with respect to potential speech communication applications; this investigation will probably be carried out for the quadratically optimal system. We hope that the practicality of the simulation and automatic design procedure we have developed for the design of optimum binary communication systems will serve as an inducement to application even in those instances where a fully analytic solution is very difficult.

## IX. EXTENSIONS TO THE BINARY MODEL

The bulk of this report has been devoted to a discussion of what we refer to as a binary communication system, and most of our work has been in this direction. However, it is possible to treat the more complex model of an M-ary digital system employing reconstruction delay by the methods we have developed, and we shall indicate the procedure in this section. It should be stated at the outset that using our procedure to determine necessary conditions no longer seems as profitable; generalizing the system model seems to have weakened the derived necessary conditions.

Referring to the block diagram of the general communication system provided in the first section, we write the system equations as follows.

$$s_n = g(I_n | \bar{I}_{n-1}) ; s_n' = S(\bar{s}_n) ; R_n = R(s_n' | \bar{s}_{n-1}') \quad (1)$$

$$\text{minimize } E(\phi(I_n - R_{n+k}) | \bar{I}_n) \\ s_n, R_{n+k}$$

We must now remember that  $\underline{s}$  represents an M-ary valued symbol rather than just a binary valued one. The only change in the form of the above equations from their previous values occurs in the error criteria where we assume a delay of  $\underline{k}$  inserted at the receiver.

The error criterion,  $E(\phi(I_n - R_{n+k}) | \bar{I}_n)$ , involves an averaging over all of the possible receiver outputs that can occur at time  $t_{n+k}$  given that  $\bar{I}_n$  has been the system input. The presence of a noisy channel generally implies that this average is to be taken over all possible receiver outputs at time  $t_{n+k}$ , and assuming that the sequence  $\bar{I}_n$  has length  $(i+1)$ , we see that there will be  $M^{(i+k+1)}$  possible different receiver outputs to be averaged over. If the channel had been noiseless, then the number of receiver

outputs to be averaged over would only have been  $M^k$ , which for zero  $k$  reduces to unity as expected.

The formal optimization procedure is now very similar to that developed in the third and fourth sections for the binary system operating without intentional delay. The essential distinction lies in the greatly increased number of alternatives now available for mutual comparison. At  $t_n$  we must decide which of  $M$  states to select for transmission or equivalently determined the  $M$  disjoint and exhaustive subsets of the real line,  $l_n(j)$  ( $j = 1, \dots, M$ ), where  $l_n(j)$  is given by:

$$l_n(j) = \left\{ I_n : E[\phi(I_n - R_{n+k}) | \bar{I}_n, s_n = j] \leq \right. \\ \left. \leq E[\phi(I_n - R_{n+k}) | \bar{I}_n, s_n = p] \text{ } p = 1, \dots, M \right. \\ \left. \text{less } j \right\} \quad (2)$$

Solutions for Eq. 2 will then yield the optimum transmitter to be used in association with a given receiver. Arguments similar to those presented in the binary case indicate that if we represent  $l_n(j)$  as a union of disjoint intervals, then the endpoints of these intervals,  $L_n(j, i_p)$ , must satisfy.

$$(\exists p \neq j) E[\phi(L_n(j, i_p) - R_{n+k}) | \bar{I}_n, s_n = j] = \\ = E[\phi(L_n(j, i_p) - R_{n+k}) | \bar{I}_n, s_n = p] \quad (3)$$

Unfortunately, Eq. 3 is only a weak and uninformative necessary condition for the determination of  $l_n(j)$ .

The optimum receiver determination proceeds precisely as it did earlier. The transmitter is assumed given, and we find by strict analogy with our earlier results that

$$E(\phi'(I_n - R_{n+k}) | \bar{s}_{n+k}) = 0 \quad (4)$$

Finally, we are required to consider the results of Eqs. 3 and 4 simultaneously, and select that pair resulting in the minimum value of the error term stated in Eq. 1.

The results stated above will become the results given in the previous section when we replace  $M$  by 2 and thereby provide a more general formulation of the digital communication problem. A special case of some interest occurs when we consider the possibility of a noiseless channel and no delay. Equations 3 and 4 then become:

$$\phi(L_n(j, i_p) - R_n(j, i_p)) = \phi(L_n(j, i_p) - R_n(p)) \text{ for some } (p \neq j), \quad (3')$$

$$E(\phi'(I_n - R_n) | \bar{s}_n) = 0. \quad (4')$$

Observe that, if we take  $\phi$  to be in  $\Phi$  or such that

$$\phi(z_1) > \phi(z_2) > -\infty \text{ if and only if } |z_1| > |z_2|,$$

then there will be a unique solution to Eq. 3', for every  $p$  given by

$$2L_n(j, p) = R_n(j) + R_n(p) \quad (5)$$

where

$$L_n(j, p) = L_n(j, i_p)$$

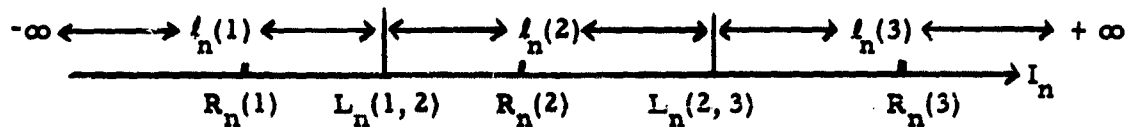
and is unique.

The result expressed by Eq. 5 contains much irrelevant information. If we return to the original inequality formulation as given by Eq. 2, we can select those solutions of Eq. 5 which are important. For convenience we order the receiver outputs as follows:

$$R_n(j) > R_n(i) \text{ if and only if } j > i. \quad (6)$$

Using Eqs. 5 and 6 in Eq. 2 leads to the conclusion that  $f_n(j)$  need only be a simple interval located between  $f_n(j-1)$  below it, and  $f_n(j+1)$  above it,

and  $l_n(j)$  contains the point  $R_n(j)$ . The upper endpoint of the interval  $l_n(j)$  bisects the interval between  $R_n(j)$  and  $R_n(j+1)$ , while the lower endpoint of  $l_n(j)$  is just the upper endpoint of  $l_n(j-1)$ ; the only exceptions to these rules are that the lower endpoint of  $l_n(1)$  is minus infinity and the upper endpoint of  $l_n(M)$  is plus infinity.



### TYPICAL TRANSMITTER ASSIGNMENT

(M = 3)

Essentially, these conclusions were obtained by Lloyd [1] under the restriction that  $\phi$  be convex and even, the channel be noiseless, and there be no reconstruction delay. Insofar as our specific results stated immediately above also require the assumption of no delay and a noiseless channel, we have only gained by solving this problem for a class of loss functions containing the convex, even loss functions as a subset. These results can be applied to the optimum M-level quantisation problem and the connection between digital communication and quantisation elucidated by means of the realization that the distribution of the random variable we wish to quantize is equivalent to the conditional distribution of the input process conditioned upon the past of the transmitted sequence; the two problems are very similar, although the digital communication problem is somewhat more general.

An extension to our work in different direction involves the consideration of the design problem for a continuous parameter input stochastic process or the attempt to transmit "analog" waveforms digitally by sampling them at times  $\{t_n\}$  and quantizing the sample values. The optimization problem may then be formulated with the same system error criterion as used previously or with a new one given by the time-weighted average of the previously employed instantaneous criterion; the time weighted criterion might take the form:

$$\text{System Error} = \int_{t_n}^{t_{n+1}} w(t, t_n) E[\phi(I(t) - R(t)) | \bar{s}_n] dt \text{ for } w \geq 0. \quad (7)$$

If we maintain the instantaneous criterion, then our earlier results generalize to cover the possibility of an analog input in a direct manner. Stating these readily verified results for the binary communication system, we have:

$$\begin{aligned} \text{Transmitter: } E[\phi(L(t_n) - R(t_n)) | s_n = 1, \bar{s}_{n-1}] &= \\ &= E[\phi(L(t_n) - R(t_n)) | s_n = -1, \bar{s}_{n-1}] \end{aligned} \quad (8)$$

$$\text{Receiver: } E[\phi(I(t) - R(t)) | \bar{s}_n] = 0 \text{ for } t_n \leq t < t_{n+1}$$

If the channel is noiseless, and  $\phi$  is in class  $\Phi$ , then Eq. 8 becomes:

$$\text{Transmitter: } 2L(t_n) = R(t_n; s_n = 1, \bar{s}_{n-1}) + R(t_n; s_n = -1, \bar{s}_{n-1}) \quad (8')$$

$$\text{Receiver: } E[\phi(I(t) - R(t)) | \bar{s}_n] = 0$$

Notice that we are only interested in  $L(t_n)$ ; it is only at time  $t_n$  that we are required to transmit in a manner chosen to optimize the reconstruction of



The design problem with the time-weighted criterion given in Eq. 7 has thus far eluded successful analysis. In our work to date the transmission problem has in part meant the subdivision of a one (for noiseless channel transmission without delay in reconstruction) or finite dimensional space into two (for binary transmission) or more regions. Now we are required to attempt the solution of this subdivision problem in an infinite dimensional function space, and it is much more difficult. Before closing, let us point out that insofar as the transmitter design is concerned, the condition that we select the transmitted state so as to select in turn the receiver output that minimizes Eq. 7 is one which is capable of direct implementation for any given input signal; one just evaluates the error term given by Eq. 7 for the given input and the two possible receiver outputs and selects that receiver output yielding the least error. It is only the problem of an initial categorization of all inputs into one of two sets that is difficult, due to the infinite dimensionality of the relevant function space for the input process sample functions.

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## APPENDIX

### A SIMULATION EXAMPLE

If we wish to construct an optimum binary communication system for a noiseless channel and a quadratic loss function given  $J$  independent sample sequences of the input process  $\{I_{nj}\}$ , then we must be able to estimate the optimum feedback loop and receiver in the equivalent delta modulation system. Our earlier work indicates that we must form estimators for the quantities below and then solve the simultaneous equations; we assume that this solution is unique, and, if not, that a search technique will inform us of this.

$$R_n = E(I_n | \bar{s}_n); L_n(\bar{s}_{n-1}) = \frac{1}{2}E(I_n | s_n = 1, \bar{s}_{n-1}) + \frac{1}{2}E(I_n | s_n = -1, \bar{s}_{n-1}) \quad (1)$$

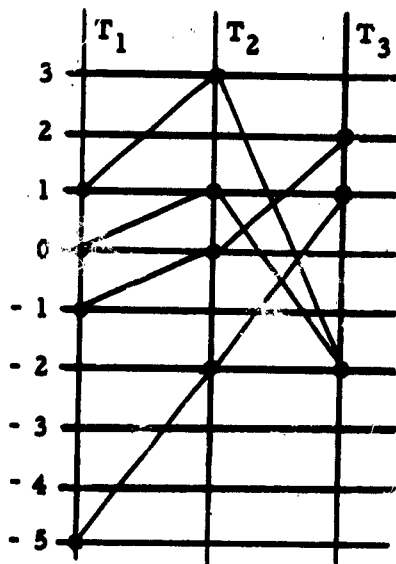
The maximum likelihood and least mean-square estimator for the conditional expectation resulting in the receiver output is given by the arithmetic average of those sample sequences at the time in question whose previous histories satisfied the conditioning on the past. The equation for  $L_n$  can also be put in a form suitable for estimation by using the receiver estimates. Thus, we have:

$$\hat{R}_n = \frac{1}{J} \sum_{j=1}^J (I_n)_j | \bar{s}_n; \hat{L}_n(\bar{s}_{n-1}) = \frac{1}{2}\hat{R}_n(s_n = 1, \bar{s}_{n-1}) + \frac{1}{2}\hat{R}_n(s_n = -1, \bar{s}_{n-1}) \quad (2)$$

For clarity of illustration, let us take a very specific example in which our dictionary consists of four sequences, each three members long. The specific sequences are given by

$$S_1(1, 3, -2); S_2(-4, -2; 1); S_3(0, 1, -2); S_4(-1, 0, 2) \quad (3)$$

The sketch below is a plot of these sequences.



At  $T_1$  we have no past information, so that  $\hat{L}_1$  is found as the average of those above plus the average of those below divided by two, where in the average all the sequences are used. We assume that the  $\hat{L}_1$  lies between  $S_3$  and  $S_4$ . Then we have:

$$\text{Ave. Above} = \left(\frac{1}{2}\right) ; \text{Ave. Below} = (-3.0) ; \text{Ave. of Both} = (-1.25) \quad (4)$$

The trial  $\hat{L}_1$  was between zero and negative unity, and is above the check value; therefore, we now move down to the next interval between adjacent samples, which is between  $S_2$  and  $S_4$ . Repeating the above, we have:

$$\text{Ave. Above} = (0) ; \text{Ave. Below} = (-5) ; \text{Ave. of Both} = (-2.5) . \quad (5)$$

The new check value of  $\hat{L}_1$ ,  $(-2.5)$ , falls in the range of the trial value which was from  $(-4)$  to  $(-2.5)$  and thus is the estimated  $L_1$ ;  $\hat{L}_1$  equals  $(-2)$ .

The receiver output corresponding to a received plus one,  $\hat{R}_1(s_1 = 1)$ , would then be given by the average above or (0) and the receiver output corresponding to a minus one,  $\hat{R}_1(s_1 = -1)$ , would be the average below or (-5).

Let us now assume that a plus one was transmitted ( $s_1 = 1$ ) and we are now at  $T_2$ . The only sequence that is to be omitted is  $S_2$ , which fell below  $\hat{L}_1$ . Thus, at this time the only remaining sample values that are acceptable, because of conditioning on the past, are (3, 1, 0). Let us assume that the trial  $\hat{L}_2(s_1 = 1)$  is between one and three, or between  $S_1$  and  $S_3$ , and calculate as above:

$$\text{Ave. Above} = (3) ; \text{Ave. Below} = \left(\frac{1}{2}\right) ; \text{Ave. Both} = \left(1\frac{3}{4}\right) \quad (6)$$

The check value falls into the same interval between samples as the trial value and, therefore,  $\hat{L}_2(1)$  equals  $1\frac{3}{4}$ . The receiver output corresponding then to ( $s_1 = 1, s_2 = 1$ ) is given by the average of those above  $\hat{L}_2(s_1 = 1)$ , and as there is only one sample  $\hat{R}_2(s_1 = 1, s_2 = 1)$  equals 3. Similarly, the receiver output for a transmitted chain ( $s_1 = 1, s_2 = -1$ ),  $\hat{R}_2(s_1 = 1, s_2 = -1)$ , would equal  $\frac{1}{2}$ .

The procedure indicated above must then be carried out for all possible transmitted binary sequences of length equal to the length of the sequences being used (in our case 3). In general, at the  $n$ th sampling time we will have to calculate  $2^{n-1} \hat{L}_n$  and  $2^n$  receiver outputs,  $\hat{R}_n$ .

Finally, if we desire to determine the error that this design would lead to in an actual application, we can proceed by taking additional samples of the input and passing them through the system just built, and determine

the resulting mean-squared errors. The average of these errors would yield an estimate for the operating error of the estimated optimum delta modulation system.

The above error includes both the estimation error and the quantization error of normal delta modulation. While this is the desirable error with regard to applications, we are still interested in determining the estimation error alone as a guide to the number of samples we would require. If we started out with  $N$  independent sample runs, we would tend to reduce the number available at the  $n$ th step by approximately a factor of  $2^n$  and could expect to have only  $N2^{-n}$  acceptable samples to use in estimating receiver and feedback loop outputs. However, since in any application we would know the exact number of samples left, let us assume that there are  $J$  samples that terminate above the level set and  $K$  samples that terminate below, and we wish to estimate the error variance in the estimation of receiver output. In particular, if we are interested in the upper reconstruction, we are concerned with the  $J$  samples. The expression for the estimation error variance,  $\mathcal{E}$ , is given by

$$\mathcal{E} = E \left\{ \left[ E(I_n | \bar{s}_n) - \frac{1}{J} \sum_j (I_n)_j | \bar{s}_n \right]^2 \right\} \quad (7)$$

where in Eq. 7 we use the unbiased estimator given by Eq. 2. Expansion of Eq. 7 and algebraic manipulation yield

$$\mathcal{E} = E^2(I_n | \bar{s}_n) + \frac{1}{J^2} \sum_{i,k} E(I_n)_i (I_n)_k | \bar{s}_n - \frac{2}{J} \sum_j E(I_n)_j | \bar{s}_n E(I_n | \bar{s}_n). \quad (8)$$

Now in Eq. 8 we make use of the independence of the samples to factor the expected value of a product of them and use the observation that the sample averages are identical with those of the random variable representing the input,  $I_n$ , to find:

$$\mathcal{E} = \frac{1}{J} \left\{ E(I_n^2 | \bar{s}_n) - E^2(I_n | \bar{s}_n) \right\} = \frac{\text{VAR}(I_n | \bar{s}_n)}{J} \quad (9)$$

Equation 9 is the desired result, as it indicates that the estimation error variance decreases as the reciprocal of the increasing number of samples,  $J$ , used in its calculation.

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